SECTION DE MATHÉMATIQUES

DR. MATHIAS BRAUN & WENHAO ZHAO

FUNCTIONAL ANALYSIS 2
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SHEET 1

Exercises for 'Functional Analysis 2' [MATH-404]

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Ex 1.1 (Some convex analysis)
Let X be a vector space. Recall that a subset $A \subset X$ is **convex** if, for all $x, y \in A$ and $\theta \in [0,1]$, it holds that $\theta x + (1-\theta)y \in A$. A function $f : A \to \mathbb{R}$ is convex if it satisfies $f(\theta x + (1-\theta)y) \leq \theta f(x) + (1-\theta) f(y)$ for all $x, y \in A$ and $\theta \in [0,1]$.

a) Show that if $(A_i)_{i\in I}$ is a family of convex subsets of X, then $A = \bigcap_{i\in I} A_i$ is also convex.

b) If $B \subset X$ is any set, we define its **convex hull** by

$$\operatorname{co}(B) = \left\{ \sum_{i=1}^{n} \lambda_i x_i : \, x_i \in B, \, \lambda_i \in [0,1], \, \sum_{i=1}^{n} \lambda_i = 1, \, n \in \mathbb{N} \right\}$$

Show that $\operatorname{co}(B)$ is the smallest convex set containing B.

c) Show that if $(f_i)_{i\in I} : A \to \mathbb{R}$ is a family of convex functions, then $f(x) := \sup_{i\in I} f_i(x)$ is convex, too.

Ex 1.2 (Properties of absorbing and balanced sets)
Let X be a vector space. Recall that a subset $A \subset X$ is :

- **absorbing** if for all $x \in X$ there exists $\varepsilon > 0$ such that for all $|t| \leq \varepsilon$ we have $tx \in A$;
- **balanced** if $\lambda A \subset A$ for all $\lambda \in \mathbb{R}$ with $|\lambda| \leq 1$.

Prove the following :

a) If $(A_i)_{i\in I}$ is a family of absorbing sets, then $\bigcup_{i\in I} A_i$ is absorbing. In addition, show that if I is finite, also $\bigcap_{i\in I} A_i$ is absorbing.

b) If $(A_i)_{i\in I}$ is a family of balanced sets, then $\bigcup_{i\in I} A_i$ and $\bigcap_{i\in I} A_i$ are balanced set.

c) Let $A \subset X$ be an arbitrary set. Show that $\bigcup_{|\lambda|\leq 1} \lambda A$ is the smallest balanced set containing A, where $\lambda \in \mathbb{R}$ (the set is called the **balanced hull**).

d) The convex hull of a balanced set is balanced.

Ex 1.3 (On convex neighborhoods of the origin in TVS)
Let (X, τ) be a topological vector space (TVS).

a) Show that the interior of a convex set is convex.

b) Show that every neighborhood of the origin is an absorbing set.

c) Show that every neighborhood of the origin contains a balanced open neighborhood of the origin.
Hint: Use the continuity of the scalar multiplication.

d) Show that every convex neighborhood of the origin contains an absorbing, balanced, convex, open neighborhood of the origin.
Hint: You might prove/use that in any TVS the convex hull of an open set is open.